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FROM Γ-SPACES TO ALGEBRAIC THEORIES

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ABSTRACT. The paper examines semi-theories, that is, formalisms of the type of the Γ-spaces of Segal which describe homotopy structures on topological spaces. It is shown that for any semi-theory one can find an algebraic theory describing the same structure on spaces as the original semi-theory. As a consequence one obtains a criterion for establishing when two semi-theories describe equivalent homotopy structures.

1. INTRODUCTION

In [13] Segal gave the following elegant characterization of infinite loop spaces. Let $\mathbf{\Gamma}^{op}$ be the category whose objects are finite sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and whose morphisms are all maps of sets $\varphi\colon [n] \to [m]$ satisfying $\varphi(0) = 0$. For $n > 1$ and $1 \leq k \leq n$ let $p^n_k\colon [n] \to [1]$ be the map such that $p^n_k(i) = 1$ only if $i = k$. A Γ-space is a functor $X\colon \mathbf{\Gamma}^{op} \to \mathbf{Spaces}$ such that $X[0]$ is a contractible space and for $n > 1$ the map $\prod_{k=1}^n X(p^n_k)\colon X[n] \longrightarrow X[1]^n$ is a weak equivalence. The main result of [13] states that giving a Γ-space amounts to describing an infinite loop spaces structure on the space $X[1]$.

The general approach of Segal's paper proved to be useful to characterize a variety of homotopy invariant structures on spaces. In preparation for his work on infinite loop spaces Segal himself showed that it is suitable for describing A_∞-spaces. Subsequently it was used by Bousfield [4] to give a characterization of n-fold loop spaces and by the author [2] to identify generalized Eilenberg–Mac Lane spaces. The formalism underlying these examples can be described as follows.

Definition 1.1. A pointed semi-theory $\mathbf{C}$ is a category with objects $[0], [1], \ldots$ such that $[0]$ is both an initial and a terminal object and for every $n > 1$ there is a distinguished ordered set of n different morphisms $p^n_1, \ldots, p^n_n \in \mathrm{Hom}_{\mathbf{C}}([n], [1])$.

A homotopy algebra over a pointed semi-theory $\mathbf{C}$ is a functor

$$X\colon \mathbf{C} \longrightarrow \mathbf{Spaces}$$

such that $X[0]$ is a contractible space and the map

$$\prod_{k=1}^n X(p^n_k)\colon X[n] \longrightarrow X[1]^n$$

is a weak equivalence for all $n > 1$.

We will call the morphisms p^n_k projection morphisms. It will be also convenient to denote by p^1_1 the identity morphism on the object $[1]$.

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Given this definition, a prototypical theorem about homotopy algebras would say that if X is a homotopy algebra over a specified semi-theory $\mathbf{C}$, then the space $X[1]$ is equipped with some structure, depending only on $\mathbf{C}$. All results mentioned above are variants of this statement.

It will be useful for us to consider also an unpointed version of Definition 1.1 where we forget about the object [0]:

Definition 1.2. An unpointed semi-theory is a category $\mathbf{C}$ with objects $[1], [2], \dots$ and for every $n \geq 1$ a choice of projection maps $p^n_1, \dots, p^n_n \in \mathrm{Hom}_{\mathbf{C}}([n], [1])$.

A homotopy algebra over $\mathbf{C}$ is a functor $X\colon \mathbf{C} \to \mathbf{Spaces}$ such that for every $n > 1$ the map $\prod_{k=1}^{n} X(p^n_k)$ gives a weak equivalence of spaces $X[n]$ and $X[1]^n$.

The major problem one faces while working with homotopy algebras is, given a specific structure on a space, how to decide which semi-theory (if any) can be used to describe it. Or, going in the opposite direction, given a semi-theory $\mathbf{C}$ how to identify the structure it puts on the space $X[1]$ of a homotopy $\mathbf{C}$-algebra X. In [1] we showed that such questions are easier to settle if one restricts attention to the following special kind of semi-theories:

Definition 1.3. A pointed (or unpointed) algebraic theory $\mathbf{T}$ is a pointed (resp. unpointed) semi-theory such that for $n > 1$ the projections in $\mathbf{T}$ induce isomorphisms

$$\prod_{k=1}^{n} p^n_k\colon \mathrm{Hom}_{\mathbf{T}}([m],[n]) \longrightarrow \mathrm{Hom}_{\mathbf{T}}([m],[1])^n.$$

Definition 1.4. Let $\mathbf{C}$ be a semi-theory. A strict $\mathbf{C}$-algebra is a functor

$$A\colon \mathbf{C} \longrightarrow \mathbf{Spaces}$$

such that the maps

$$\prod_{k=1}^{n} A(p^n_k)\colon A[n] \longrightarrow A[1]^n$$

are isomorphisms for all $n > 1$. If $\mathbf{C}$ is pointed, then we also assume that $A[0] = *$.

The main result of [1] states that if $\mathbf{T}$ is an algebraic theory, then every homotopy $\mathbf{T}$-algebra can be replaced by a weakly equivalent strict $\mathbf{T}$-algebra. On the other hand, giving a strict $\mathbf{T}$-algebra A amounts to providing the space $A[1]$ with some sort of an algebraic structure determined by $\mathbf{T}$, e.g., monoid, group, ring, Lie algebra, ... (see [10],[3]). Therefore, homotopy algebras over algebraic theories essentially describe spaces weakly equivalent to algebraic objects in **Spaces**.

In the present paper we show that it is possible to replace homotopy algebras by strict algebras even if one works with a semi-theory which is not an algebraic theory (as it is the case with Segal's Γ^{op}) as follows.

Theorem 1.5. *For any pointed (or unpointed) semi-theory* $\mathbf{C}$ *there exists a pointed (resp. unpointed) simplicial algebraic theory* $\overline{\mathbf{F}_*\mathbf{C}}$ *such that the homotopy category of homotopy* $\mathbf{C}$*-algebras is equivalent to the homotopy category of strict* $\overline{\mathbf{F}_*\mathbf{C}}$*-algebras. Moreover, the construction of* $\overline{\mathbf{F}_*\mathbf{C}}$ *is functorial in* $\mathbf{C}$.

By a simplicial algebraic theory $\mathbf{T}$ we mean a simplicial object in the category of algebraic theories. Equivalently, $\mathbf{T}$ is a category enriched over simplicial sets such that the projection morphisms are in the dimension zero of the simplicial sets $\mathrm{Hom}_{\mathbf{T}}([m],[1])$ and induce isomorphisms of simplicial sets as in Definition 1.3. Simplicial semi-theories are defined analogously.

As an application of Theorem 1.5 we can formulate conditions when two semi-theories describe equivalent structures on spaces.

Theorem 1.6. *A functor of semi-theories $G\colon \mathbf{C} \longrightarrow \mathbf{C}'$ induces an equivalence of the homotopy categories of homotopy algebras iff the induced functor $G\colon \overline{\mathbf{F}_*\mathbf{C}} \to \overline{\mathbf{F}_*\mathbf{C}'}$ between the associated simplicial algebraic theories is a weak equivalence of categories; that is, G is a weak equivalence on the simplicial sets of morphisms of these categories.*

Relationship to the theorem of Dwyer and Kan. Let $\mathbf{C}$ be a small category and let $\mathbf{D}$ be its subcategory. Denote by $\mathbf{Spaces}^{\mathbf{C},\mathbf{D}}$ the category of all diagrams $X\colon \mathbf{C} \to \mathbf{Spaces}$ such that for every morphism $\varphi \in \mathbf{D}$ the map $X(\varphi)$ is a weak equivalence. Let $\mathbf{D}' \subseteq \mathbf{C}'$ be another pair of categories and assume that we have a functor $G\colon (\mathbf{C},\mathbf{D}) \to (\mathbf{C}',\mathbf{D}')$ which is an epimorphism on the sets of objects. In [5] Dwyer and Kan gave sufficient and necessary conditions for the functor G which guarantee that the induced functor $G^*\colon \mathbf{Spaces}^{\mathbf{C}',\mathbf{D}'} \to \mathbf{Spaces}^{\mathbf{C},\mathbf{D}}$ yields an equivalence of the homotopy categories of diagrams. Their approach involves constructing for a pair $(\mathbf{C},\mathbf{D})$ a simplicial category $\mathbf{F}_*\mathbf{C}[\mathbf{F}_*\mathbf{D}]$. Since this construction is functorial, given G as above, we obtain a new functor

$$FG\colon \mathbf{F}_*\mathbf{C}[\mathbf{F}_*\mathbf{D}] \longrightarrow \mathbf{F}_*\mathbf{C}'[\mathbf{F}_*\mathbf{D}'].$$

Then G^* yields an equivalence of the homotopy categories of diagrams if and only if the functor FG is a weak equivalence of categories.

We note that our construction of the category $\overline{\mathbf{F}_*\mathbf{C}}$ is very analogous to the construction of $\mathbf{F}_*\mathbf{C}[\mathbf{F}_*\mathbf{D}]$. This is not very surprising since homotopy algebras resemble the diagrams considered by Dwyer and Kan: a homotopy $\mathbf{C}$-algebra X is a diagram of spaces satisfying the condition that certain maps defined in terms of $\mathbf{C}$ are weak equivalences. The difference is that in our setting the maps which are weak equivalences are not images of morphisms of $\mathbf{C}$. From this perspective one can view Theorem 1.6 as an extension of [5, 2.5].

Organization of the paper. Throughout this paper we will work with unpointed versions of Theorems 1.5 and 1.6. The main difference between the pointed and the unpointed case is that the notion of a free unpointed semi-theory (Section 3) is more natural and as a consequence the construction of the algebraic theory $\overline{\mathbf{F}_*\mathbf{C}}$ is easier to describe. In Section 9 we comment on the changes required in order to perform this construction in the pointed case. Beyond that our arguments work for pointed semi-theories without any major changes. Hence, from now on by semi-theory we will understand an unpointed semi-theory (and similarly for algebraic theories).

We use freely the language of model categories. Section 2 contains a concise presentation of the main model category structures we will need. In Section 3, we describe a functorial construction which associates to every semi-theory $\mathbf{C}$ an algebraic theory $\bar{\mathbf{C}}$ (which we call the completion of $\mathbf{C}$) in such a way, that the categories of strict algebras over $\mathbf{C}$ and $\bar{\mathbf{C}}$ are canonically isomorphic. This construction, however, usually does not preserve the category of homotopy algebras. To deal with this problem in Section 4 we replace the semi-theory $\mathbf{C}$ with $\mathbf{F}_*\mathbf{C}$—the simplicial resolution of $\mathbf{C}$. This resolution is a simplicial semi-theory. Applying the completion of §3 to every simplicial dimension of $\mathbf{F}_*\mathbf{C}$ we obtain a simplicial algebraic theory $\overline{\mathbf{F}_*\mathbf{C}}$. Then the results of Section 2 show that the proof of Theorem 1.5 amounts to verifying that certain mapping complexes of $\mathbf{F}_*\mathbf{C}$–diagrams are weakly equivalent. Assuming this we give the proof of Theorem 1.6. The following two sections contain a few technical facts needed to complete the proof of Theorem 1.5. In Section 5 we describe some properties of homotopy algebras over a semi-theory $\mathbf{P}$ which is the initial object in the category of semi-theories. Then, in §6, we deal with the problem of describing the space of maps between diagrams indexed by a simplicial category in terms of maps of diagrams over discrete categories. Finally, in Sections 7 and 8 we complete the proof of Theorem 1.5.

Notation. (i) This paper is written simplicially. Thus, **Spaces** stands for the category of simplicial sets and consequently by "space" we always mean a simplicial set.

(ii) For a small category $\mathbf{C}$ by $\mathbf{Spaces}^{\mathbf{C}}$ we will denote the category of all functors $\mathbf{C} \to \mathbf{Spaces}$. If X, Y are objects of $\mathbf{Spaces}^{\mathbf{C}}$, then by $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ we will mean the set of all natural transformations $X \to Y$. Since objects of the indexing category $\mathbf{C}$ will be $[0], [1], \ldots$ (except for §6) this notation should not lead to confusing $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ with $\mathrm{Hom}_{\mathbf{C}}([n], [m])$—the set of morphisms $[n] \to [m]$ in $\mathbf{C}$.

2. Model categories

Below we describe model category structures which we will use throughout this paper. Our setting is basically the same as in [1], therefore we discuss it briefly and refer there for the details.

Model category for homotopy algebras. Let $\mathbf{C}$ be a small category. We consider two model category structures on $\mathbf{Spaces}^{\mathbf{C}}$, denoted by $\mathbf{Spaces}^{\mathbf{C}}_{fib}$ and $\mathbf{Spaces}^{\mathbf{C}}_{cof}$ [1, sec. 3]. Weak equivalences in both cases are objectwise weak equivalences. Fibrations in $\mathbf{Spaces}^{\mathbf{C}}_{fib}$ are objectwise fibrations and cofibrations in $\mathbf{Spaces}^{\mathbf{C}}_{cof}$ are objectwise cofibrations. The third class of morphisms in each of these model categories is as usual determined by the above choices. Both $\mathbf{Spaces}^{\mathbf{C}}_{fib}$ and $\mathbf{Spaces}^{\mathbf{C}}_{cof}$ are simplicial model categories and they share the same simplicial structure: for $X \in \mathbf{Spaces}^{\mathbf{C}}$ and a simplicial set K the functor $X \otimes K$ is defined by $X \otimes K(c) = X(c) \times K$ for all $c \in \mathbf{C}$.

For $X, Y \in \mathbf{Spaces}^{\mathbf{C}}$ by $\mathrm{Map}_{\mathbf{C}}(X, Y)$ we will denote the simplicial function complex of X and Y, that is the simplicial set whose k–dimensional simplices are all natural transformations $X \otimes \Delta[k] \to Y$.

If $\mathbf{C}$ is a semi-theory, then similarly as in [1, sec. 5] we get an additional model category structure on $\mathbf{Spaces}^{\mathbf{C}}$ which we will denote by $\mathbf{LSpaces}^{\mathbf{C}}$. Cofibrations in $\mathbf{LSpaces}^{\mathbf{C}}$ are the same as cofibrations in $\mathbf{Spaces}^{\mathbf{C}}_{fib}$. A map $f\colon X \to Y$ is a weak equivalence in $\mathbf{LSpaces}^{\mathbf{C}}$ if for every homotopy $\mathbf{C}$-algebra Z which is fibrant as an object of $\mathbf{Spaces}^{\mathbf{C}}_{cof}$ the induced map of simplicial mapping complexes

$$f^*\colon \mathrm{Map}_{\mathbf{C}}(Y, Z) \longrightarrow \mathrm{Map}_{\mathbf{C}}(X, Z)$$

is a weak equivalence. To distinguish such maps from objectwise weak equivalences we will call them local equivalences.

The significance of the model category $\mathbf{LSpaces}^{\mathbf{C}}$ for our purposes lies in the following fact [1, 5.7]:

Proposition 2.1. *The homotopy category of* $\mathbf{LSpaces}^{\mathbf{C}}$ *is equivalent to the category obtained by taking the full subcategory of* $\mathbf{Spaces}^{\mathbf{C}}$ *spanned by homotopy* $\mathbf{C}$*-algebras and inverting all objectwise weak equivalences.*

Thus, $\mathbf{LSpaces}^{\mathbf{C}}$ is a model category suitable for describing the homotopy theory of homotopy $\mathbf{C}$-algebras.

The category of strict algebras. Let $\mathbf{C}$ be again a semi-theory and let $\mathbf{Alg}^{\mathbf{C}}$ denote the full subcategory of $\mathbf{Spaces}^{\mathbf{C}}$ whose objects are all strict $\mathbf{C}$-algebras. We have the inclusion functor $J_{\mathbf{C}}\colon \mathbf{Alg}^{\mathbf{C}} \longrightarrow \mathbf{Spaces}^{\mathbf{C}}$. Arguments analogous to these of [1, 2.4] imply

Proposition 2.2. *There exists a functor*

$$K_{\mathbf{C}}\colon \mathbf{Spaces}^{\mathbf{C}} \longrightarrow \mathbf{Alg}^{\mathbf{C}}$$

left adjoint to $J_{\mathbf{C}}$*.*

Using arguments similar to these of [14, 3.1] one can show that $\mathbf{Alg}^{\mathbf{C}}$ has a simplicial model category structure with objectwise weak equivalences and objectwise fibrations. Using this fact we get [1, 6.3]

Proposition 2.3. *The adjoint pair of functors* $(K_{\mathbf{C}}, J_{\mathbf{C}})$ *is a Quillen pair between model categories* $\mathbf{LSpaces}^{\mathbf{C}}$ *and* $\mathbf{Alg}^{\mathbf{C}}$*.*

Comparison lemma. The main step in the proof of Theorem 1.5 is to demonstrate that for certain semi-theories $\mathbf{C}$ the Quillen pair of 2.3 is a Quillen equivalence (which implies that the homotopy theories of homotopy $\mathbf{C}$-algebras and strict $\mathbf{C}$-algebras are equivalent). The next lemma shows that this in turn amounts to verifying that certain maps in $\mathbf{LSpaces}^{\mathbf{C}}$ are local equivalences. For a semi-theory $\mathbf{C}$ denote by $C_n \in \mathbf{Spaces}^{\mathbf{C}}$ the functor corepresented by $[n]$; that is,

$$C_n[m] := \mathrm{Hom}_{\mathbf{C}}([n], [m]).$$

Let $\eta_{C_n}\colon C_n \longrightarrow J_{\mathbf{C}}K_{\mathbf{C}}C_n$ be the unit of the adjunction $(K_{\mathbf{C}}, J_{\mathbf{C}})$.

Lemma 2.4. *If the map* η_{C_n} *is a local equivalence for all* $n \geq 0$*, then the Quillen pair* $(K_{\mathbf{C}}, J_{\mathbf{C}})$ *is a Quillen equivalence of the model categories* $\mathbf{LSpaces}^{\mathbf{C}}$ *and* $\mathbf{Alg}^{\mathbf{C}}$*.*

Proof. The arguments used in the proof of [1, 6.4] show that $(K_{\mathbf{C}}, J_{\mathbf{C}})$ is a Quillen equivalence if $\eta_X\colon X \longrightarrow J_{\mathbf{C}}K_{\mathbf{C}}X$ is a local equivalence for any cofibrant object $X \in \mathbf{LSpaces}^{\mathbf{C}}$. By assumption the map η_X is a local equivalence if $X = C_n$. We claim that η_X is a local equivalence also if X is a finite disjoint sum of such functors, $X = \coprod_i^m C_{n_i}$. Indeed, notice that for $m \geq 1$ we have a map

$$\coprod_{j=1}^{m} C_1 \longrightarrow C_m$$

which is induced by the projections $p^m_j \in \mathbf{C}$. We can use it to construct a commutative diagram

$$\begin{array}{ccc}
\coprod_{i=1}^m C_{n_i} & \xrightarrow{\eta} & J_{\mathbf{C}}K_{\mathbf{C}}(\coprod_{i=1}^n C_{n_i}) \\
\uparrow{\scriptstyle f} & & \uparrow{\scriptstyle J_{\mathbf{C}}K_{\mathbf{C}}(f)} \\
\coprod_{i=1}^m \coprod_{j=1}^{n_i} C_1 & \xrightarrow{\eta} & J_{\mathbf{C}}K_{\mathbf{C}}(\coprod_{i=1}^m \coprod_{j=1}^{n_i} C_1) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle J_{\mathbf{C}}K_{\mathbf{C}}(g)} \\
C_{\sum n_i} & \xrightarrow{\eta} & J_{\mathbf{C}}K_{\mathbf{C}}(C_{\sum n_i})
\end{array}$$

Directly from the definition of a local equivalence it follows that both left vertical maps are local equivalences. One can also check that if A is a strict $\mathbf{C}$-algebra, then the maps induced by f and g on sets $\mathrm{Hom}_{\mathbf{C}}(-, A)$ are isomorphisms. This implies that the maps $J_{\mathbf{C}}K_{\mathbf{C}}(f)$ and $J_{\mathbf{C}}K_{\mathbf{C}}(g)$ are isomorphisms. By our assumption the bottom map η is a local equivalence. Thus, also the top map must be a local equivalence. This proves our claim.

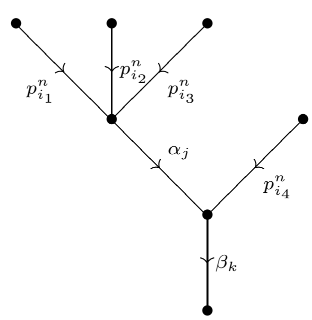
In order to show that η_X is a local equivalence for an arbitrary cofibrant object X one can proceed from here using the same arguments as in the proof of [1, 6.5]. □

3. Completion of semi-theories

Definition 3.1. A free semi-theory is a semi-theory $\mathbf{C}$ which is free as a category, and whose projection morphisms are among free generators of $\mathbf{C}$.

Our goal in this section is to describe a functorial construction which associates to every free semi-theory $\mathbf{C}$ an algebraic theory $\bar{\mathbf{C}}$ such that the categories of strict algebras over $\mathbf{C}$ and over $\bar{\mathbf{C}}$ are isomorphic.

The category $\bar{\mathbf{C}}$ is defined as follows. The objects of $\bar{\mathbf{C}}$ are the same as the objects of $\mathbf{C}$: $\mathrm{ob}\bar{\mathbf{C}} = \mathrm{ob}\mathbf{C} = \{[1], [2], \dots\}$. For $n \geq 1$ the set of morphisms $\mathrm{Hom}_{\bar{\mathbf{C}}}([n], [1])$ consists of directed trees T:



satisfying the following conditions:

1) the lowest vertex of T has only one incoming edge;
2) all edges of T are labeled with symbols α_i where $\alpha\colon [m] \longrightarrow [k]$ is some generator of $\mathbf{C}$ and $1 \leq i \leq k$ (if $\alpha = p^n_k$, then we will write p^n_k rather than $(p^n_k)_1$);
3) if a vertex of T has m incoming edges, then the outgoing edge is labeled with α_i for some $\alpha\colon [m] \longrightarrow [k]$;

4) all the initial edges (that is, the edges starting at vertices with no incoming edges) are labeled with projections p^n_k and no other edge of T is labeled with any projection morphism.

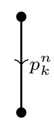
For $m > 1$ we define

$$\mathrm{Hom}_{\bar{\mathbf{C}}}([n],[m]) := \mathrm{Hom}_{\bar{\mathbf{C}}}([n],[1])^m$$

If $(T_1, \ldots, T_m) \in \mathrm{Hom}_{\bar{\mathbf{C}}}([n],[m])$ and $S \in \mathrm{Hom}_{\bar{\mathbf{C}}}([m],[1])$, then the composition $S \circ (T_1, \ldots, T_m)$ is a tree obtained by grafting the tree T_i in place of each initial edge of S labeled with p^m_i. In general, if $(T_1, \ldots, T_m) \in \mathrm{Hom}_{\bar{\mathbf{C}}}([n],[m])$ and $(S_1, \ldots, S_k) \in \mathrm{Hom}_{\bar{\mathbf{C}}}([m],[k])$, then

$$(S_1, \ldots, S_k) \circ (T_1, \ldots, T_m) := (S_1 \circ (T_1, \ldots, T_m), \ldots, S_k \circ (T_1, \ldots, T_m))$$

The identity morphism $\mathrm{id}_{[n]} \in \mathrm{Hom}_{\bar{\mathbf{C}}}([n],[n])$ is given by $\mathrm{id}_{[n]} = (\mathbf{p}^n_1, \ldots, \mathbf{p}^n_n)$ where $\mathbf{p}^n_k$ is the tree

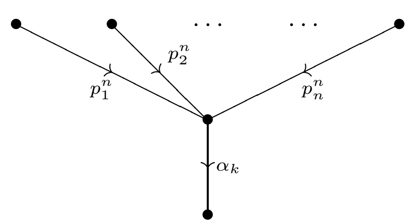


We give $\bar{\mathbf{C}}$ a semi-theory structure by choosing the morphisms $\mathbf{p}^n_k$ to be projections in $\bar{\mathbf{C}}$. Directly from the construction it follows that $\bar{\mathbf{C}}$ is an algebraic theory.

Next we define a functor

$$\Phi_{\mathbf{C}} \colon \mathbf{C} \longrightarrow \bar{\mathbf{C}}$$

which is the identity on objects, and such that $\Phi_{\mathbf{C}}(p^n_k) = \mathbf{p}^n_k$. If $\alpha \colon [n] \longrightarrow [m]$ is a generator of $\mathbf{C}$ and α is not a projection, then $\Phi_{\mathbf{C}}(\alpha) = (T_{\alpha_1}, \ldots, T_{\alpha_m})$ where T_{α_k} is the tree



Definition 3.2. We call the functor $\Phi_{\mathbf{C}}$ the completion of the semi-theory $\mathbf{C}$ to an algebraic theory.

As we mentioned at the beginning of this section the essential property of the completion is that it preserves the category of strict algebras over $\mathbf{C}$:

Proposition 3.3. *For any semi-theory* $\mathbf{C}$ *the functor* $\Phi_{\mathbf{C}} \colon \mathbf{C} \longrightarrow \bar{\mathbf{C}}$ *induces an isomorphism of the categories of strict algebras*

$$\Phi^*_{\mathbf{C}} \colon \mathbf{Alg}^{\bar{\mathbf{C}}} \xrightarrow{\cong} \mathbf{Alg}^{\mathbf{C}}.$$

The proof will use the following

Definition 3.4. Let $T \in \mathrm{Hom}_{\bar{\mathbf{C}}}([n],[1])$. By $l(T)$ we will denote the number of edges of the longest (directed) path contained in T. If $T = (T_1, \ldots, T_m) \in \mathrm{Hom}_{\bar{\mathbf{C}}}([n],[m])$, then $l(T) := \max\{l(T_1), \ldots, l(T_m)\}$.

Proof of Proposition 3.3. We will construct a functor

$$\Psi_{\mathbf{C}}\colon \mathbf{Alg}^{\mathbf{C}} \longrightarrow \mathbf{Alg}^{\bar{\mathbf{C}}}$$

which will be the inverse of $\Phi^*_{\mathbf{C}}$. Let $X\colon \mathbf{C} \longrightarrow \mathbf{Spaces}$ be a strict $\mathbf{C}$–algebra. Set

$$\Psi_{\mathbf{C}}X[n] := X[n].$$

For $n \geq 1$ denote by $\varphi_n\colon X[n] \longrightarrow X[1]^n$ the isomorphism induced by the product of projections, $\varphi_n := \prod_k X(p^n_k)$. Let T be a morphism of $\bar{\mathbf{C}}$. We define $\Psi_{\mathbf{C}}X(T)$ by induction with respect to $l(T)$:

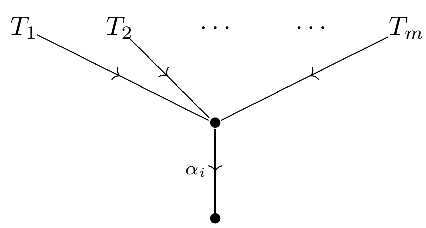
1) If $T\colon [n] \longrightarrow [1]$ and $l(T) = 1$, then $T = \mathbf{p}^n_k$ for some $1 \leq k \leq n$. Then

$$\Psi_{\mathbf{C}}X(\mathbf{p}^n_k) := X(p^n_k).$$

2) Assume that $\Psi_{\mathbf{C}}X(T)$ has been already defined for all $T\colon [n] \longrightarrow [1]$ such that $l(T) \leq r$ and let $(T_1, \dots, T_m) \in \mathrm{Hom}_{\bar{\mathbf{C}}}([n], [m])$, $l(T_1, \dots, T_m) \leq r$. Then for $x \in \Psi_{\mathbf{C}}X[n]$ set

$$\Psi_{\mathbf{C}}X(T_1, \dots, T_m)(x) := \varphi^{-1}_m(\Psi_{\mathbf{C}}X(T_1)(x), \dots, \Psi_{\mathbf{C}}X(T_m)(x)).$$

3) Let $\Psi_{\mathbf{C}}X(T)$ be defined for all morphisms T with $l(T) \leq r$ and let $S \in \mathrm{Hom}_{\bar{\mathbf{C}}}([n], [1])$, $l(S) = r + 1$. Then S is of the form



where $(T_1, \dots, T_m)\colon [n] \longrightarrow [m]$ is a morphism with $l(T_1, \dots, T_m) = r$, $\alpha\colon [m] \longrightarrow [k]$ is a generator of $\mathbf{C}$ and $1 \leq i \leq k$. In this case

$$\Psi_{\mathbf{C}}X(S) := X(p^m_i \circ \alpha) \circ \Psi_{\mathbf{C}}X(T_1, \dots, T_m).$$

One can check that $\Psi_{\mathbf{C}}$ is a well-defined functor and that the compositions $\Psi_{\mathbf{C}} \circ \Phi^*_{\mathbf{C}}$ and $\Phi^*_{\mathbf{C}} \circ \Psi_{\mathbf{C}}$ are both identities. □

Remark 3.5. While we will not use it, we note that completion of a semi-theory can be described in categorical terms as follows. Let **AlgTh** and **SemiTh** denote the categories of algebraic theories and semi-theories respectively. We have the forgetful functor

$$R\colon \mathbf{AlgTh} \longrightarrow \mathbf{SemiTh}.$$

One can show that there exists a functor $L\colon \mathbf{SemiTh} \longrightarrow \mathbf{AlgTh}$ left adjoint to R. For a semi-theory $\mathbf{C}$ let $\eta_{\mathbf{C}}\colon \mathbf{C} \longrightarrow RL\mathbf{C}$ denote the unit of this adjunction. If $\mathbf{C}$ is a free semi-theory, then we have $\bar{\mathbf{C}} = RL\mathbf{C}$ and $\Phi_C = \eta_{\mathbf{C}}$. Moreover, the functor induced by $\eta_{\mathbf{C}}$ on the categories of strict algebras is an isomorphism for any (not necessarily free) semi-theory $\mathbf{C}$.

4. Simplicial resolution of a semi-theory

Let $\mathbf{C}$ be a semi-theory. Following [7, 2.5] we denote by $\mathbf{F}_*\mathbf{C}$ the simplicial resolution of $\mathbf{C}$. Thus, $\mathbf{F}_*\mathbf{C}$ is a simplicial category such that $\mathbf{F}_0\mathbf{C}$ is a free category whose generators are morphisms of $\mathbf{C}$ and, for $k > 0$, $\mathbf{F}_k\mathbf{C}$ is a free category generated by $\mathbf{F}_{k-1}\mathbf{C}$. For every $[m], [n] \in \mathbf{C}$ there is a canonical map

$$\psi_{n,m}\colon \mathrm{Hom}_{\mathbf{C}}([n],[m]) \longrightarrow \mathrm{Hom}_{\mathbf{F}_*\mathbf{C}}([n],[m]).$$

We can define a simplicial semi-theory structure on $\mathbf{F}_*\mathbf{C}$ by choosing projection morphisms in $\mathbf{F}_*\mathbf{C}$ to be the images of projections of $\mathbf{C}$ under the maps $\psi_{n,1}$. We also have a functor [7, 2.5]

$$\Psi\colon \mathbf{F}_*\mathbf{C} \longrightarrow \mathbf{C}.$$

Since for any $[m], [n]$ the composition

$$\mathrm{Hom}_{\mathbf{C}}([n],[m]) \xrightarrow{\psi_{n,m}} \mathrm{Hom}_{\mathbf{F}_*\mathbf{C}}([n],[m]) \xrightarrow{\Psi} \mathrm{Hom}_{\mathbf{C}}([n],[m])$$

is the identity map, the functor Ψ is a map of semi-theories. Moreover, the following holds:

Proposition 4.1. *The functor Ψ induces an adjoint pair of functors between model categories of homotopy algebras*

$$\Psi_*\colon \mathbf{LSpaces}^{\mathbf{F}_*\mathbf{C}} \rightleftarrows \mathbf{LSpaces}^{\mathbf{C}}\colon \Psi^*$$

which is a Quillen equivalence.

Proof. By [7, 2.6] the functor Ψ is a weak equivalence of the categories $\mathbf{C}$ and $\mathbf{F}_*\mathbf{C}$. It follows that the Quillen pair induced by Ψ between the categories of diagrams

$$\Psi_*\colon \mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{fib} \rightleftarrows \mathbf{Spaces}^{\mathbf{C}}_{fib}\colon \Psi^*$$

is a Quillen equivalence (this is essentially a consequence of [7, 2.1]). The model categories $\mathbf{LSpaces}^{\mathbf{C}}$ and $\mathbf{LSpaces}^{\mathbf{F}_*\mathbf{C}}$ are obtained by localizing $\mathbf{Spaces}^{\mathbf{C}}_{fib}$ and $\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{fib}$, and so the statement follows from [9, Thm. 3.3.20] which states that localization preserves Quillen equivalences. □

Since for every $k \geq 0$ the category $\mathbf{F}_k\mathbf{C}$ is a free semi-theory, we can construct its completion to an algebraic theory (Definition 3.2)

$$\Phi_k\colon \mathbf{F}_k\mathbf{C} \longrightarrow \overline{\mathbf{F}_k\mathbf{C}}.$$

The functors Φ_k can be combined to define a functor of simplicial categories

$$\Phi\colon \mathbf{F}_*\mathbf{C} \longrightarrow \overline{\mathbf{F}_*\mathbf{C}}$$

where $\overline{\mathbf{F}_*\mathbf{C}}$ is a simplicial algebraic theory which has $\overline{F_k\mathbf{C}}$ in its k-th simplicial dimension. Using 3.3 we get

Lemma 4.2. *The functor $\Phi\colon \mathbf{F}_*\mathbf{C} \longrightarrow \overline{\mathbf{F}_*\mathbf{C}}$ induces an isomorphism of categories of strict algebras*

$$\Phi^*\colon \mathbf{Alg}^{\overline{\mathbf{F}_*\mathbf{C}}} \xrightarrow{\cong} \mathbf{Alg}^{\mathbf{F}_*\mathbf{C}}.$$

Recall (Proposition 2.3) that we have a Quillen pair of functors

$$K_{\mathbf{F}_*\mathbf{C}}\colon \mathbf{LSpaces}^{\mathbf{F}_*\mathbf{C}} \rightleftarrows \mathbf{Alg}^{\mathbf{F}_*\mathbf{C}}\colon J_{\mathbf{F}_*\mathbf{C}}\,.$$

In order to prove Theorem 1.5 it is enough to show that the following holds.

Proposition 4.3. *The Quillen pair* $(K_{\mathbf{F}_*\mathbf{C}}, J_{\mathbf{F}_*\mathbf{C}})$ *is a Quillen equivalence.*

Proof of Theorem 1.5. By Proposition 4.1 the homotopy category of homotopy $\mathbf{C}$-algebras is equivalent to the homotopy category of homotopy $\mathbf{F}_*\mathbf{C}$-algebras. Using in turn Proposition 4.3 we get that this last homotopy category is equivalent to the homotopy category of strict $\mathbf{F}_*\mathbf{C}$-algebras. The isomorphism of Lemma 4.2 completes the proof. □

The statement of Proposition 4.3 is an immediate consequence of Lemma 2.4 and

Lemma 4.4. *For* $n \geq 0$ *let* $FC_n \in \mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}$ *denote the functor corepresented by* $[n] \in \mathbf{F}_*\mathbf{C}$*. Then the unit of adjunction of the pair* $(K_{\mathbf{F}_*\mathbf{C}}, J_{\mathbf{F}_*\mathbf{C}})$*,*

$$\eta_{FC_n} \colon FC_n \longrightarrow J_{\mathbf{F}_*\mathbf{C}} K_{\mathbf{F}_*\mathbf{C}} FC_n,$$

is a local equivalence.

The proof of Lemma 4.4 requires some technical preparations; we postpone it until §7. Meanwhile we turn to the proof of Theorem 1.6. It will use the following fact (see also [12, 8.6]):

Lemma 4.5. *Let* $F \colon \mathbf{T} \to \mathbf{T}'$ *be a functor of simplicial algebraic theories. Then* F *induces an adjoint pair of functors between the categories of strict algebras*

$$F_* \colon \mathbf{Alg}^{\mathbf{T}} \rightleftarrows \mathbf{Alg}^{\mathbf{T}'} \colon F^*$$

which is a Quillen pair. Moreover, F^* *gives an equivalence of the homotopy theories of strict algebras iff the functor* F *is a weak equivalence of categories.*

Proof. For the existence of the adjoint pair (F_*, F^*) see [3, 3.7.7]. It is a Quillen pair by [14, 3.4]. Also by [14, 3.4] if F is a weak equivalence of $\mathbf{T}$ and $\mathbf{T}'$, then (F_*, F^*) is a Quillen equivalence. Therefore, it remains to show that if F^* induces an equivalence of the homotopy categories, then F is a weak equivalence.

Let T_n denote the $\mathbf{T}$-diagram corepresented by $[n] \in \mathbf{T}$. Since $\mathbf{T}$ is an algebraic theory, T_n is a strict $\mathbf{T}$-algebra. Moreover, T_n is a cofibrant object in $\mathbf{Alg}^{\mathbf{T}}$. The functor F^* gives an equivalence of the homotopy categories of strict algebras, and thus its left adjoint F_* provides the inverse equivalence. Therefore, the unit of the adjunction (F_*, F^*),

$$\mu \colon T_n \longrightarrow F^* F_* T_n,$$

is an (objectwise) weak equivalence in $\mathbf{Alg}^{\mathbf{T}}$. For $m \geq 0$ the map $\mu_{[m]} \colon T_n[m] \to F^* F_* T_n[m]$ is given by maps of simplicial sets of morphisms

$$T_n[m] = \mathrm{Hom}_{\mathbf{T}}([n],[m]) \xrightarrow{F} \mathrm{Hom}_{\mathbf{T}'}([n],[m]) = F^* F_* T_n[m].$$

But this just means that $F \colon \mathbf{T} \to \mathbf{T}'$ is a weak equivalence of categories. □

Proof of Theorem 1.6. Let $G \colon \mathbf{C} \to \mathbf{C}'$ be a functor of semi theories. Consider the diagram

$$\begin{array}{ccc}
\mathbf{Alg}^{\overline{\mathbf{F}_*\mathbf{C}}} & \rightleftarrows & \mathbf{Alg}^{\overline{\mathbf{F}_*\mathbf{C}}'} \\
\upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright \\
\mathbf{LSpaces}^{\mathbf{F}_*\mathbf{C}} & \rightleftarrows & \mathbf{LSpaces}^{\mathbf{F}_*\mathbf{C}'} \\
\upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright \\
\mathbf{LSpaces}^{\mathbf{C}} & \rightleftarrows & \mathbf{LSpaces}^{\mathbf{C}'}
\end{array}$$

in which every pair of arrows represents a Quillen pairs of functors. The horizontal pairs are induced by the functor G while the vertical ones come from the adjunctions of 4.1, 2.3 and 4.2. Propositions 4.1 and 4.3 imply that the vertical pairs are in fact Quillen equivalences. Therefore, G induces an equivalence of the homotopy categories of $\mathbf{LSpaces}^{\mathbf{C}}$ and $\mathbf{LSpaces}^{\mathbf{C}'}$ if and only if it induces an equivalence of the homotopy categories of strict algebras $\mathbf{Alg}^{\overline{\mathbf{F}_*\mathbf{C}}}$ and $\mathbf{Alg}^{\overline{\mathbf{F}_*\mathbf{C}}'}$. Thus Lemma 4.5 completes the proof. □

5. The initial semi-theory

In order to complete our argument we still need to prove Lemma 4.4. In this and the following section we develop some technical tools we will use. The proof of the lemma itself will be given in §7.

Denote by $\mathbf{P}$ the semi-theory whose only non-identity morphisms are projections. This is the initial object in the category of semi-theories: for any semi-theory $\mathbf{C}$ there is a unique map of semi-theories $\mathbf{P} \to \mathbf{C}$.

The following fact states that local equivalences in $\mathbf{Spaces}^{\mathbf{P}}$ are particularly easy to detect.

Proposition 5.1. *A map $f\colon X \to Y$ in $\mathbf{Spaces}^{\mathbf{P}}$ is a local equivalence if and only if the restriction $f_1\colon X[1] \to Y[1]$ is a weak equivalence of spaces.*

In the proof it will be convenient to use a definition of a local equivalence which is different (but equivalent, see [9, 18.6.31]) to the one given in §2. First, assume that X, Y are cofibrant in $\mathbf{Spaces}^{\mathbf{P}}_{fib}$. Then a map $f\colon X \to Y$ is a local equivalence if for every homotopy algebra Z fibrant in $\mathbf{Spaces}^{\mathbf{P}}_{fib}$ the induced map of homotopy function complexes

$$f^*\colon \mathrm{Map}_{\mathbf{P}}(Y, Z) \longrightarrow \mathrm{Map}_{\mathbf{P}}(X, Z)$$

is a weak equivalence of simplicial sets. If X and Y are not cofibrant, then the map f is a local equivalence if the induced map $f'\colon X' \to Y'$ between cofibrant replacements of X and Y is one.

Proof of Lemma 5.1. Without loss of generality, we can assume that X and Y are cofibrant in $\mathbf{Spaces}^{\mathbf{P}}_{fib}$. Observe that for any strict $\mathbf{P}$-algebra Z we have an isomorphism

$$\mathrm{Map}_{\mathbf{P}}(X, Z) \simeq \mathrm{Map}(X[1], Z[1])$$

where the space on the right-hand side is a mapping complex of simplicial sets. Moreover, given any Kan complex K one can construct a strict $\mathbf{P}$-algebra Z_K which is fibrant in $\mathbf{Spaces}^{\mathbf{P}}_{fib}$ and such that $Z_K[1] = K$. It easily follows from here that if $f\colon X \to Y$ is a local equivalence, then $f_1\colon X[1] \to Y[1]$ is a weak equivalence.

To see that the second implication holds, notice that for any homotopy algebra $Z \in \mathbf{Spaces}^{\mathbf{P}}$ one can find a strict $\mathbf{P}$-algebra Z' such that Z' is fibrant in $\mathbf{Spaces}^{\mathbf{P}}_{fib}$ and there is an objectwise weak equivalence $Z \to Z'$. □

In Section 7 we will use Proposition 5.1 to detect local equivalences between diagrams over a semi-theory $\mathbf{F}_*\mathbf{C}$ (§4). In order to achieve that we will need

Lemma 5.2. *The map of semi-theories $J\colon \mathbf{P} \to \mathbf{F}_*\mathbf{C}$ induces a Quillen pair of functors*

$$J_*\colon \mathbf{Spaces}^{\mathbf{P}}_{cof} \rightleftarrows \mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof} \colon J^* .$$

The proof will follow from

Lemma 5.3. *Let* $J_{\mathbf{C}}\colon \mathbf{P} \to \mathbf{C}$ *denote the inclusion of* $\mathbf{P}$ *into a free semi-theory* $\mathbf{C}$*. Then the adjoint pair of functors*

$$J_{\mathbf{C}*}\colon \mathbf{Spaces}^{\mathbf{P}}_{cof} \rightleftarrows \mathbf{Spaces}^{\mathbf{C}}_{cof}\colon {J_{\mathbf{C}}}^*$$

is a Quillen pair

Proof. The functor $J_{\mathbf{C}*}\colon \mathbf{Spaces}^{\mathbf{P}} \to \mathbf{Spaces}^{\mathbf{C}}$ can be described as follows. If $Y \in \mathbf{Spaces}^{\mathbf{P}}$ and $[n] \in \mathbf{C}$, then

$$J_{\mathbf{C}*}Y[n] = Y[n] \sqcup \coprod_{(\varphi\colon [m]\to[n])\in C_n} Y[m]$$

where C_n is the set of all morphisms $\varphi\colon [m] \to [n]$ such that $\varphi = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1$ where $\alpha_1, \dots, \alpha_k$ are generators of $\mathbf{C}$ and α_1 is not a projection.

From this description it is clear that $J_{\mathbf{C}*}$ preserves objectwise cofibrations and weak equivalences. Therefore, $(J_{\mathbf{C}*}, {J_{\mathbf{C}}}^*)$ is a Quillen pair. □

Proof of Lemma 5.2. For every $k = 0, 1, \dots$ the functor J defines a map of semi-theories

$$J_k\colon \mathbf{P} \longrightarrow \mathbf{F}_k\mathbf{C}$$

Let $s\mathbf{Spaces}$ denote the category of simplicial spaces taken with the model structure in which cofibrations and weak equivalences are the objectwise cofibrations and weak equivalences. The functors

$$(J_k)_*\colon \mathbf{Spaces}^{\mathbf{P}}_{cof} \longrightarrow \mathbf{Spaces}^{F_k\mathbf{C}}_{cof}$$

can be assembled into a functor

$$J_\bullet\colon \mathbf{Spaces}^{\mathbf{P}}_{cof} \longrightarrow s\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}$$

such that

$$J_\bullet Y[n] = ((J_0)_* Y[n] \Longleftarrow (J_1)_* Y[n] \Lleftarrow \dots)\ .$$

Take the diagonal functor

$$|-|\colon s\mathbf{Spaces} \longrightarrow \mathbf{Spaces}$$

which associates to every simplicial space $X_\bullet$ its diagonal $|X_\bullet|$. It induces a functor

$$|-|\colon s\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof} \longrightarrow \mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}.$$

One can check that $J_*\colon \mathbf{Spaces}^{\mathbf{P}}_{cof} \to \mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}$ is equal to the composition

$$\mathbf{Spaces}^{\mathbf{P}}_{cof} \xrightarrow{J_\bullet} s\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof} \xrightarrow{|-|} \mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}.$$

Since $\mathbf{F}_k\mathbf{C}$ is a free semi-theory for all $k \geq 0$, Lemma 5.3 implies that $J_\bullet$ sends cofibrations and weak equivalences from $\mathbf{Spaces}^{\mathbf{P}}_{cof}$ to objectwise weak equivalences and cofibrations in $s\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}$. The diagonal functor, in turn sends such maps to weak equivalences and cofibrations in $\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}$. Therefore, the functor $J_*\colon \mathbf{Spaces}^{\mathbf{P}}_{cof} \to \mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}$ preserves cofibrations and weak equivalences and the adjoint pair (J_*, J^*) is a Quillen pair as claimed. □

6. Mapping complexes of diagrams over a simplicial category

Let $\mathbf{C}$ be an arbitrary simplicial category. Denote by $\mathbf{C}_k$ the category in k-th simplicial dimension of $\mathbf{C}$. Our goal in this section is to show that given $X, Y \in \mathbf{Spaces}^{\mathbf{C}}$ one can describe the simplicial mapping complex $\mathrm{Map}_{\mathbf{C}}(X,Y)$ by means of mapping complexes of diagrams over the discrete categories $\mathbf{C}_k$.

Thus, assume that X is a diagram of spaces over $\mathbf{C}$. We consider two ways in which we can turn X into a diagram over the category $\mathbf{C}_k$:

1) For $c \in \mathbf{C}_k$ let $X_k(c)$ be the set of k–dimensional simplices of $X(c)$ (regarded as a discrete simplicial set). If $\varphi\colon c \to d$ is a morphism of $\mathbf{C}_k$, then the functor X gives a map $X_k(c) \to X_k(d)$. Thus we obtain a diagram $X_k \in \mathbf{Spaces}^{\mathbf{C}_k}$. One can check that the correspondence $X \mapsto X_k$ defines a functor $R_k\colon \mathbf{Spaces}^{\mathbf{C}} \to \mathbf{Spaces}^{\mathbf{C}_k}$.
2) Let $\varphi\colon c \to d$ again be a morphism of $\mathbf{C}_k$. The diagram X provides a map
$$X(\varphi)\colon X(c) \times \Delta[k] \longrightarrow X(d).$$
This can be used to define a functor $L_kX\colon \mathbf{C}_k \longrightarrow \mathbf{Spaces}$ such that $L_kX(c) := X(c) \times \Delta[k]$ and for $\varphi\colon c \to d$ in $\mathbf{C}_k$ the map $L_kX(\varphi)$ is given by
$$L_kX(\varphi)\colon X(c) \times \Delta[k] \xrightarrow{X(\varphi)\times pr_2} X(d) \times \Delta[k]$$
where pr_2 is the projection on the second factor. Again, we observe that this construction is functorial in X and yields a functor $L_k\colon \mathbf{Spaces}^{\mathbf{C}} \to \mathbf{Spaces}^{\mathbf{C}_k}$.

Let $\theta\colon k \to l$ be a morphism in the simplicial indexing category Δ^{op}. Since $\mathbf{C}$ is a simplicial category, we have a functor $\theta\colon \mathbf{C}_k \to \mathbf{C}_l$ which induces
$$\theta^*\colon \mathbf{Spaces}^{\mathbf{C}_l} \longrightarrow \mathbf{Spaces}^{\mathbf{C}_k}.$$
For any $X \in \mathbf{Spaces}^{\mathbf{C}}$ there are obvious natural transformations of $\mathbf{C}_k$–diagrams:
$$f_\theta\colon X_k \longrightarrow \theta^* X_l$$
and
$$g_\theta\colon \theta^* L_l X \longrightarrow L_k X.$$
Now, for $X, Y \in \mathbf{Spaces}^{\mathbf{C}}$ consider the function complex $\mathrm{Map}_{\mathbf{C}_k}(X_k, L_kY)$. A morphism $\theta \in \Delta^{op}$ as above defines a map
$$\mathrm{Map}_{\mathbf{C}_l}(X_l, L_lY) \longrightarrow \mathrm{Map}_{\mathbf{C}_l}(\theta^* X_l, \theta^* L_l Y).$$
Composing it with the maps induced by f_θ and g_θ yields a map
$$\theta^*\colon \mathrm{Map}_{\mathbf{C}_l}(X_l, L_lX) \longrightarrow \mathrm{Map}_{\mathbf{C}_k}(X_k, L_kY).$$
It is not hard to check that it defines a cosimplicial space $\mathrm{Map}_{\mathbf{C}_\bullet}(X, LY)$ which has $\mathrm{Map}_{\mathbf{C}_k}(X_k, L_kY)$ in k-th cosimplicial dimension. We claim that the simplicial function complex $\mathrm{Map}_{\mathbf{C}}(X,Y)$ can be recovered from this cosimplicial space. Recall [8, VIII.1, p. 390] that to every cosimplicial space $Z_\bullet$ one can associate its total space $\mathrm{Tot}(Z_\bullet)$.

Proposition 6.1. *For any simplicial category* $\mathbf{C}$ *and* $X, Y \in \mathbf{Spaces}^{\mathbf{C}}$ *the simplicial sets* $\mathrm{Map}_{\mathbf{C}}(X,Y)$ *and* $\mathrm{Tot}\,\mathrm{Map}_{\mathbf{C}_\bullet}(X, LY)$ *are isomorphic. Moreover, the isomorphism is functorial in* X *and* Y.

Proof. The total space $\operatorname{Tot}\operatorname{Map}_{\mathbf{C}_\bullet}(X, LY)$ is the equalizer of the diagram

$$\prod_{k\geq 0}\operatorname{Map}_{\mathbf{C}_k}(X_k\times\Delta[k], L_kY) \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} \prod_{\phi\colon k\to l}\operatorname{Map}_{\mathbf{C}_k}(X_k\times\Delta[l], L_kY)$$

where the second product is indexed by all morphisms $\phi\in\Delta^{op}$. We claim that there exists a map

$$\psi\colon \operatorname{Map}_{\mathbf{C}}(X,Y)\longrightarrow\prod_{k\geq 0}\operatorname{Map}_{\mathbf{C}_k}(X_k\times\Delta[k], L_kY)$$

such that $\delta_1\circ\psi=\delta_2\circ\psi$. As a consequence there is a unique map $\bar{\psi}\colon\operatorname{Map}_{\mathbf{C}}(X,Y)\to\operatorname{Tot}\operatorname{Map}_{\mathbf{C}_\bullet}(X, LY)$ which fits into the following commutative diagram:

$$\begin{array}{ccc}\operatorname{Map}_{\mathbf{C}}(X,Y) & & \\ \downarrow{\scriptstyle\bar{\psi}} & \overset{\psi}{\searrow} & \\ \operatorname{Tot}\operatorname{Map}_{\mathbf{C}_\bullet}(X, LY) & \longrightarrow & \prod_{k\geq 0}\operatorname{Map}_{\mathbf{C}_k}(X_k\times\Delta[k], L_kY)\end{array}$$

The map ψ is defined as follows. For $k\geq 0$ let $\epsilon_k\colon X_k\times\Delta[k]\to L_kX$ be a natural transformation given by

$$\epsilon_k = i_k\times pr_2\colon X_k(c)\times\Delta[k]\longrightarrow X(c)\times\Delta[k]$$

where i_k is the inclusion of the set of k-dimensional simplices into $X(c)$ and pr_2 is the projection on the second factor. Consider the induced map of simplicial sets

$$\epsilon_k^*\colon\operatorname{Map}_{\mathbf{C}_k}(L_kX, L_kY)\longrightarrow\operatorname{Map}_{\mathbf{C}_k}(X_k\times\Delta[k], L_kY).$$

Its composition with the map $\operatorname{Map}_{\mathbf{C}}(X,Y)\to\operatorname{Map}_{\mathbf{C}_k}(L_kX, L_kY)$ given by the functor L_k yields

$$\psi_k\colon\operatorname{Map}_{\mathbf{C}}(X,Y)\longrightarrow\operatorname{Map}_{\mathbf{C}_k}(X_k\times\Delta[k], L_kY).$$

We define $\psi := \prod_{k\geq 0}\psi_k$.

One can check that ψ is an embedding of simplicial sets and thus so is $\bar{\psi}$. It is also not hard to see that $\bar{\psi}$ is an epimorphism. Therefore, it gives an isomorphism of $\operatorname{Map}_{\mathbf{C}}(X,Y)$ and $\operatorname{Tot}\operatorname{Map}_{\mathbf{C}_\bullet}(X, LY)$. □

7. Proof of Lemma 4.4

Let $\mathbf{D}$ be a free semi-theory and $\Phi_{\mathbf{D}}\colon\mathbf{D}\longrightarrow\bar{\mathbf{D}}$ the completion of $\mathbf{D}$ to an algebraic theory. Denote by D_n the $\mathbf{D}$-diagram corepresented by $[n]$. Using the functor $\Phi_{\mathbf{D}}$ we can also define a $\mathbf{D}$-diagram $\bar{D}_n$ such that $\bar{D}_n[m]=\operatorname{Hom}_{\bar{\mathbf{D}}}([n],[m])$, and a map of $\mathbf{D}$-diagrams

$$\Phi_{\mathbf{D}}^*\colon D_n\longrightarrow\bar{D}_n.$$

Moreover, since $\Phi_{\mathbf{D}}$ is an embedding of categories, D_n is a subdiagram of $\bar{D}_n$. We define a filtration of the diagram $\bar{D}_n$ by $\mathbf{D}$-diagrams

$$\bar{D}_n^0\subseteq\bar{D}_n^1\subseteq\ldots\bar{D}_n$$

as follows. Set $\bar{D}_n^0 := D_n$. If $\bar{D}_n^i$ is defined for $i\leq k$, then $\bar{D}_n^{k+1}$ is the smallest $\mathbf{D}$-subdiagram of $\bar{D}_n$ such that if $T_1, T_2, \ldots, T_m$ are elements of $\bar{D}_n^k[1]\subseteq\operatorname{Hom}_{\bar{\mathbf{D}}}([n],[1])$, then $(T_1, T_2, \ldots, T_m)\in\operatorname{Hom}_{\bar{\mathbf{D}}}([n],[m])$ belongs to $\bar{D}_n^{k+1}[m]$. From the definition of $\bar{\mathbf{D}}$ (§3) it follows that $\operatorname{colim}_k\bar{D}_n^k=\bar{D}_n$.

Recall (§5) that by $\mathbf{P}$ we denoted the semi-theory which has projections as the only non-identity morphisms. The unique map $\mathbf{P} \to \mathbf{D}$ induces a $\mathbf{P}$-diagram structure on D_n, $\bar{D}_n$ and $\bar{D}_n^k$. Similarly, as above we define a filtration of $\bar{D}_n$ by $\mathbf{P}$-diagrams

$$s\bar{D}_n^0 \subseteq s\bar{D}_n^1 \subseteq \cdots \subseteq \bar{D}_n$$

where $s\bar{D}_n^0 = D_n$ and $s\bar{D}_n^{k+1}$ is the smallest $\mathbf{P}$-subdiagram of $\bar{D}_n$ such that if $T_1, T_2, \ldots, T_m \in \bar{D}_n^k[1]$, then $(T_1, T_2, \ldots, T_m) \in s\bar{D}_n^{k+1}[m]$. We have inclusions of $\mathbf{P}$-diagrams $\bar{D}_n^k \subseteq s\bar{D}_n^{k+1} \subseteq \bar{D}_n^{k+1}$ and $\operatorname{colim}_k s\bar{D}_n = \bar{D}_n$.

We claim that the filtrations $\{\bar{D}_n^k\}$ and $\{s\bar{D}_n^k\}$ have the following property:

Lemma 7.1. *For any* $\mathbf{D}$*-diagram of spaces* $X\colon \mathbf{D} \longrightarrow \mathbf{Spaces}$ *and for* $k \geq 0$ *the square of simplicial mapping complexes*

$$\begin{array}{ccc} \operatorname{Map}_{\mathbf{D}}(\bar{D}_n^k, X) & \longleftarrow & \operatorname{Map}_{\mathbf{D}}(\bar{D}_n^{k+1}, X) \\ \downarrow & & \downarrow \\ \operatorname{Map}_{\mathbf{P}}(\bar{D}_n^k, X) & \longleftarrow & \operatorname{Map}_{\mathbf{P}}(s\bar{D}_n^{k+1}, X) \end{array}$$

is a pullback diagram.

The proof of this fact is given in Section 8.

Now we can proceed with the proof of Lemma 4.4. Consider $\mathbf{F}_m\mathbf{C}$, the free semi-theory in the m-th simplicial dimension of $\mathbf{F}_*\mathbf{C}$, and let $\overline{\mathbf{F}_m\mathbf{C}}$ be its completion to an algebraic theory. Setting $\mathbf{D} := \mathbf{F}_m\mathbf{C}$ above we see that the $\mathbf{F}_m\mathbf{C}$-diagram $\overline{F_mC}_n$, (where $\overline{F_mC}_n[r] = \operatorname{Hom}_{\overline{\mathbf{F}_m\mathbf{C}}}([n],[r])$) admits two filtrations: by $\mathbf{F}_m\mathbf{C}$-diagrams

$$F_mC_n = \overline{F_mC}_n^0 \subseteq \overline{F_mC}_n^1 \subseteq \cdots \subseteq \overline{F_mC}_n$$

and by $\mathbf{P}$-diagrams

$$F_mC_n = s\overline{F_mC}_n^0 \subseteq s\overline{F_mC}_n^1 \subseteq \cdots \subseteq \overline{F_mC}_n.$$

The first of these filtrations, combined for all m, yields a filtration of the diagram $\overline{FC}_n$ by $\mathbf{F}_*\mathbf{C}$-diagrams

$$FC_n = \overline{FC}_n^0 \subseteq \overline{FC}_n^1 \subseteq \cdots \subseteq \overline{FC}_n.$$

Similarly, the filtrations of $\overline{F_mC}$ by $\mathbf{P}$-diagrams $s\overline{F_mC}_n^k$ for $m \geq 0$ give a filtration of $\overline{FC}_n$ by $\mathbf{P}$-diagrams

$$FC_n = s\overline{FC}_n^0 \subseteq s\overline{FC}_n^1 \subseteq \cdots \subseteq \overline{FC}_n.$$

Lemma 7.2. *For* $X \in \mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}$ *consider the following diagrams of simplicial function complexes:*

$$\begin{array}{ccc} \operatorname{Map}_{\mathbf{F}_*\mathbf{C}}(\overline{FC}_n^k, X) & \xleftarrow{\ f\ } & \operatorname{Map}_{\mathbf{F}_*\mathbf{C}}(\overline{FC}_n^{k+1}, X) \\ \downarrow & & \downarrow \\ \operatorname{Map}_{\mathbf{P}}(\overline{FC}_n^k, X) & \xleftarrow{\ g\ } & \operatorname{Map}_{\mathbf{P}}(s\overline{FC}_n^{k+1}, X) \end{array}$$

This is a pullback diagram for all X*, and* $k, n \geq 0$*.*

Proof. Take the diagrams of cosimplicial spaces (Section 6)

$$\begin{array}{ccc}
\mathrm{Map}_{\mathbf{F}_\bullet\mathbf{C}}(\overline{FC}^k_n, LX) & \xleftarrow{f_\bullet} & \mathrm{Map}_{\mathbf{F}_\bullet\mathbf{C}}(\overline{FC}^{k+1}_n, LX) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathbf{P}}(\overline{FC}^k_n, LX) & \xleftarrow{g_\bullet} & \mathrm{Map}_{\mathbf{P}}(s\overline{FC}^{k+1}_n, LX)
\end{array}$$

Since limits of cosimplicial spaces can be computed by taking limits in each cosimplicial dimension separately, Lemma 7.1 implies that this is a pullback diagram of cosimplicial spaces. Therefore, our claim is a consequence of 6.1 and of the fact that the functor Tot commutes with limits. □

Consider the map g in the statement of Lemma 7.2. Our next goal is

Lemma 7.3. *Let X be a homotopy algebra fibrant in $\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}$. Then for every $k \geq 0$ the map*

$$g\colon \mathrm{Map}_{\mathbf{P}}(s\overline{FC}^{k+1}_n, X) \longrightarrow \mathrm{Map}_{\mathbf{P}}(\overline{FC}^k_n, X)$$

is an acyclic fibration of simplicial sets.

Proof. The map g is induced by an inclusion $\iota_k\colon \overline{FC}^k_n \hookrightarrow s\overline{FC}^{k+1}_n$. Since all inclusions are cofibrations in $\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}$ we get that g is a fibration. Thus, it remains to show that g is a weak equivalence of simplicial sets.

By Lemma 5.2 if X a homotopy $\mathbf{F}_*\mathbf{C}$-algebra fibrant in $\mathbf{Spaces}^{\mathbf{F}_*\mathbf{C}}_{cof}$, then it is also a $\mathbf{P}$-homotopy algebra which is fibrant in $\mathbf{Spaces}^{\mathbf{P}}_{cof}$. Therefore, it is enough to show that the map ι_k is a local equivalence in $\mathbf{Spaces}^{\mathbf{P}}$. This, however, is a consequence of 5.1 and the observation that ι_k restricts to an isomorphism of simplicial sets

$$\overline{FC}^k_n[1] \xrightarrow{\cong} s\overline{FC}^{k+1}_n[1].$$

□

Next, consider the upper map f in the diagram in Lemma 7.2. Combining 7.3 and 7.2 we obtain

Corollary 7.4. *For all $n, k \geq 0$ the map*

$$f\colon \mathrm{Map}_{\mathbf{F}_*\mathbf{C}}(\overline{FC}^{k+1}_n, X) \longrightarrow \mathrm{Map}_{\mathbf{F}_*\mathbf{C}}(\overline{FC}^k_n, X)$$

is an acyclic fibration of simplicial sets.

We use this fact to finish the proof of Lemma 4.4. The map $\eta_{FC_n}\colon FC_n \to J_{\mathbf{C}}K_{\mathbf{C}}FC_n$ is given by the inclusion of $\mathbf{F}_*\mathbf{C}$-diagrams $FC_n = \overline{FC}^0_n \hookrightarrow \overline{FC}_n$. Moreover, $\overline{FC}_n$ is represented by the colimit $\overline{FC}_n = \mathrm{colim}_k\, \overline{FC}^k_n$. We have a commutative diagram

$$\begin{array}{ccc}
FC_n & \longrightarrow & \mathrm{hocolim}_k \overline{FC}^k_n \\
& {\scriptstyle\eta_{FC_n}}\searrow \quad \swarrow & \\
& \mathrm{colim}_k\, \overline{FC}^k_n &
\end{array}$$

where the homotopy colimit is taken in the model category $\mathbf{LSpaces}^{\mathbf{F}_*\mathbf{C}}$. Corollary 7.4 implies that both maps $FC_n \to \mathrm{hocolim}_k \overline{FC}^k_n$ and $\mathrm{hocolim}_k \overline{FC}^k_n \to$

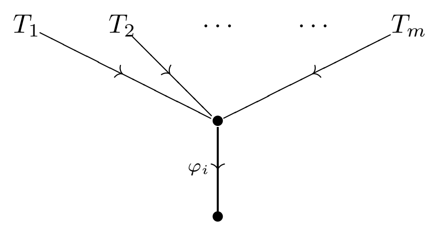
$\operatorname{colim}_k \overline{FC}_n^k$ are local equivalences. Therefore, the map η_{FC_n} is also a local equivalence.

8. The pullback lemma

In order to prove Lemma 7.1 we need to make a few observations about $\bar{D}_n$ and its filtrations. For the rest of this section α_i will always denote a generator of the free semi-theory $\mathbf{D}$. In particular, if φ is a morphism of $\mathbf{D}$, then by $\varphi = \alpha_k \circ \cdots \circ \alpha_1$ we will understand the decomposition of φ into generators of $\mathbf{D}$.

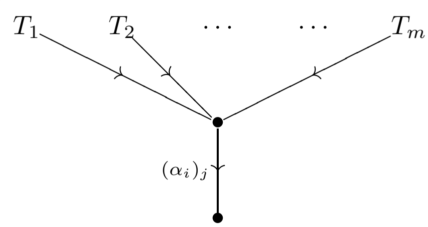
Lemma 8.1. *Let $\varphi = \alpha_k \circ \cdots \circ \alpha_1$ be a morphism of $\mathbf{D}$ such that α_1 is not a projection morphism. If $T, T' \in \bar{D}_n$ satisfy $\varphi(T) = \varphi(T')$, then $T = T'$.*

Proof. Assume that $\varphi = \alpha_1 : [m] \longrightarrow [r]$ is a generator of $\mathbf{D}$. If $T \in \bar{D}_n[m]$, then $T = (T_1, \dots, T_m)$ where $T_1, \dots, T_m$ are trees contained in $\operatorname{Hom}_{\bar{\mathbf{D}}}([n], [1])$. Using the definition of the composition of morphisms in $\bar{\mathbf{D}}$ we get that $\varphi(T) = (\varphi_1 T, \dots, \varphi_k T)$ where $\varphi_i T$ is a tree in $\bar{\mathbf{D}}$ of the form



Since $\varphi_i T$ contains all information about T, the equality $\varphi T = \varphi T'$ must imply $T = T'$.

Next, assume that $\varphi = \alpha_2 \circ \alpha_1$ where $\alpha_1 \colon [m] \longrightarrow [r]$ and α_2 is a projection morphism, say $\alpha_2 = p_j^r$. In this case $\varphi(T)$ is the tree $(\alpha_1)_j T$:



Thus, as before, $\varphi T = \varphi T'$ implies that $T = T'$.

To show that the lemma holds for an arbitrary $\varphi = \alpha_k \circ \cdots \circ \alpha_1$ one can now argue by induction with respect to k. □

The next fact is a relative version of 8.1.

Lemma 8.2. *Let $T, T' \in \bar{D}_n$ and $\varphi = \alpha_k \circ \cdots \circ \alpha_1$, $\varphi' = \alpha'_{k'} \circ \cdots \circ \alpha'_1$ be morphisms of $\mathbf{D}$ such that $k \leq k'$ and α_1, α'_1 are not projections. If $\varphi(T) = \varphi(T')$, then $\varphi' = \varphi \circ \theta$ and $T = \theta(T')$ where $\theta = \alpha_{k'-k} \circ \cdots \circ \alpha_1$.*

Proof. We use induction with respect to k. If $k = 0$, then $\varphi = \mathrm{id}$ and the claim obviously holds. Assume that $k \geq 0$ and that α_k is not a projection. Directly inspecting trees as in the proof of 8.1 one sees that the equality $\varphi(T) = \varphi(T')$ implies $\alpha_{k'} = \alpha_k$. Therefore, we have

$$\alpha_k \circ (\alpha_{k-1} \circ \cdots \circ \alpha_1)(T) = \varphi(T) = \varphi'(T') = \alpha_k \circ (\alpha'_{k'-1} \circ \cdots \circ \alpha'_1)(T').$$

By Lemma 8.1 we get from here that

$$(\alpha_{k-1} \circ \cdots \circ \alpha_1)(T) = (\alpha'_{k'-1} \circ \cdots \circ \alpha'_1)(T')$$

and the inductive hypothesis implies that $(\alpha'_{k-1} \circ \cdots \circ \alpha'_1) = (\alpha_{k-1} \circ \cdots \circ \alpha_1) \circ \theta$ and $T = \theta(T')$ for $\theta = \alpha'_{k'-k} \circ \cdots \circ \alpha'_1$. Therefore, also $\varphi' = \varphi \circ \theta$.

It remains to consider the case when α_k is a projection. By our assumption this is possible only if $k > 1$. Moreover, α_{k-1} cannot be a projection. One can check directly that $\alpha'_{k'} = \alpha_k$ and $\alpha'_{k'-1} = \alpha_{k-1}$, and then argue the same way as above to complete the proof. □

Lemma 8.3. *Let $T \in s\bar{D}^{k+1}_n \setminus \bar{D}^k_n$ and let $\varphi = \alpha_m \circ \cdots \circ \alpha_1$ be a morphism in $\mathbf{D}$ such that α_1 is not a projection. Then $\varphi(T) \notin \bar{D}^k_n$.*

Proof. Notice that by the definition of $\bar{D}^k_n$ and $s\bar{D}^k_n$ if S is an element of $\bar{D}^k_n$, then $S = \psi(S')$ for some $\psi \in \mathbf{D}$ and $S' \in s\bar{D}^k_n$. We can also assume that $\psi = \alpha'_l \circ \cdots \circ \alpha'_1$ where α'_1 is not a projection.

Assume that for some $T \in s\bar{D}^{k+1}_n \setminus \bar{D}^k_n$ and for some $\varphi \in \mathbf{D}$ we have $\varphi(T) \in \bar{D}^k_n$. Then $\varphi(T) = \varphi'(T')$ for some $\varphi' \in \mathbf{D}$ and $T' \in s\bar{D}^k_n$. Thus by 8.2 either $T' = \theta T$ or $T = \theta(T')$ for some morphism $\theta \in \mathbf{D}$. The latter is impossible: indeed, we would get $T = \theta(T') \in \bar{D}^k_n$ contrary to the assumption we made about T. Therefore, $T' = \theta(T)$ and we can also assume that the decomposition of θ into generators of $\mathbf{D}$ does not start with a projection. Replacing θ with $p \circ \theta$ if necessary (where p is a projection morphism) we get that $\theta(T) \in \bar{D}^{k-1}_n$. Continuing with this argument we would eventually have to conclude that there exists a morphism $\theta' = \alpha'_l \circ \cdots \circ \alpha'_1 \in \mathbf{D}$ such that α'_1 is not a projection morphism and $\theta'(T)$ belongs to $\bar{D}^0_n$. But $\bar{D}^0_n = D_n$, so $\theta'(T)$ would have to be represented by a morphism of $\mathbf{D}$. One can check that this can happen only if T itself belongs to D_n. But $D_n \subseteq \bar{D}^k_n$, so this last statement contradicts our assumption on T. □

Lemma 8.4. *Let $T \in \bar{D}^{k+1}_n \setminus s\bar{D}^{k+1}_n$. There exists a unique element $S \in s\bar{D}^{k+1}_n \setminus \bar{D}^k_n$ and a unique morphism $\varphi \in \mathbf{D}$ such that $\varphi(S) = T$.*

Proof. Assume that $T = \varphi(S) = \varphi'(S')$ for some $\varphi, \varphi' \in \mathbf{D}$ and some $S, S' \in s\bar{D}^{k+1}_n \setminus \bar{D}^k_n$. Since decompositions of φ and φ' into generators of $\mathbf{D}$ cannot start with a projection, by 8.2 we can assume that $S = \theta(S')$ and $\varphi' = \varphi \circ \theta$ for some morphism $\theta \in \mathbf{D}$. It follows that $\theta(S') \in s\bar{D}^{k+1}_n$ and composing θ with some projection morphism p we have $p \circ \theta(S') \in \bar{D}^k_n$. Lemma 8.3 implies that the decomposition of $p \circ \theta$ into generators of $\mathbf{D}$ must begin with a projection. Therefore, θ must be an identity morphism. It follows that $S = S'$ and $\varphi = \varphi'$. □

Finally, we can turn to

Proof of Lemma 7.1. We will show that for any $\mathbf{D}$-diagram X the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{D}}(\bar{D}^k_n, X) & \longleftarrow & \mathrm{Hom}_{\mathbf{D}}(\bar{D}^{k+1}_n, X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{P}}(\bar{D}^k_n, X) & \longleftarrow & \mathrm{Hom}_{\mathbf{P}}(s\bar{D}^{k+1}_n, X) \end{array}$$

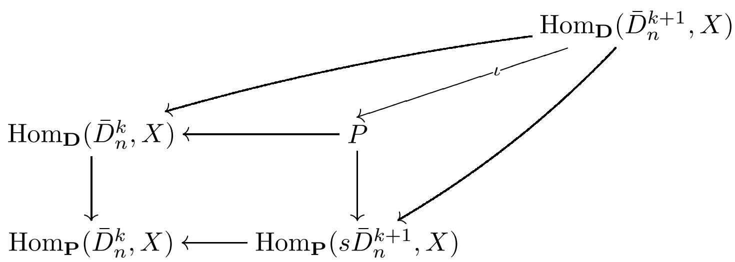
is a pullback diagram of sets. Since limits of diagrams of simplicial sets can be calculated by taking the limit in each simplicial dimension separately, and since for

$Y, X \in \mathbf{Spaces}^{\mathbf{D}}$, the set of k-dimensional simplices $\mathrm{Map}_{\mathbf{D}}(Y, X)$ can be described as $\mathrm{Map}_{\mathbf{D}}(Y, X)_m = \mathrm{Hom}_{\mathbf{D}}(Y, X^{\Delta[m]})$. The statement of Lemma 7.1 will follow.

Let P denote the set which is the pullback of the diagram

$$\mathrm{Hom}_{\mathbf{D}}(\bar{D}^k_n, X) \longrightarrow \mathrm{Hom}_{\mathbf{P}}(\bar{D}^k_n, X) \longleftarrow \mathrm{Hom}_{\mathbf{P}}(s\bar{D}^{k+1}_n, X).$$

The commutativity of the square diagram above implies that there exists a unique map $\iota\colon \mathrm{Hom}_{\mathbf{D}}(\bar{D}^{k+1}_n, X) \to P$ such that the following diagram commutes:



We want to construct the inverse of the map ι. Notice that the elements of P are maps of $\mathbf{P}$-diagrams $\epsilon\colon s\bar{D}^{k+1}_n \to X$ such that $\epsilon|_{\bar{D}^k_n}\colon \bar{D}^k_n \to X$ commutes with all morphisms of $\mathbf{D}$. We claim that such ϵ admits a unique extension to a map of $\mathbf{D}$-diagrams $\bar{\epsilon}\colon \bar{D}^{k+1}_n \to X$. To see that take $T \in \bar{D}^{k+1}_n \setminus s\bar{D}^{k+1}_n$. By 8.4 there exists a unique element $S \in s\bar{D}^{k+1}_n$ and $\varphi \in \mathbf{D}$ such that $\varphi(S) = T$. Define $\bar{\epsilon}(T) := \varphi(\epsilon(S))$. Uniqueness of this extension is obvious, and using 8.3 and 8.4 it is also not difficult to check that $\bar{\epsilon}$ is a well-defined map of $\mathbf{D}$-diagrams. The correspondence $\epsilon \mapsto \bar{\epsilon}$ gives a map $\nu\colon P \longrightarrow \mathrm{Hom}_{\mathbf{D}}(\bar{D}^{k+1}_n, X)$. One can verify that the compositions $\nu \circ \iota$ and $\iota \circ \nu$ are both identities. Therefore, we get $P \cong \mathrm{Hom}_{\mathbf{D}}(\bar{D}^{k+1}_n, X)$. □

9. The pointed case

We describe a few changes one needs to make in order to prove Theorems 1.5 and 1.6 for pointed semi-theories. Since, as we noted in §1, the arguments we used for unpointed semi-theories apply in this case with minor changes only, we will concentrate on the differences in some definitions and constructions.

First, since free categories usually do not have terminal and initial objects, in the pointed case Definition 3.1 must be modified as follows:

Definition 9.1. A pointed semi-theory $\mathbf{C}$ is free if

– there exists a free subcategory $\mathbf{C}_{>0} \subset \mathbf{C}$ with objects $[1], [2], \dots$ such that all projections in $\mathbf{C}$ are free generators of $\mathbf{C}_{>0}$;

– the only morphisms of $\mathbf{C}$ which do not belong to $\mathbf{C}_{>0}$ are the morphims $[n] \to [0]$, $[0] \to [n]$ for $n \geq 0$ and their compositions.

In this setting by generators of a free pointed semi-theory $\mathbf{C}$ we understand the generators of $\mathbf{C}_{>0}$.

Notice that the category $\mathbf{C}_{>0}$ is a free unpointed semi-theory. Going in the opposite direction, given any free unpointed semi-theory $\mathbf{C}$ one can construct a unique free pointed semi-theory $\mathbf{C}_+$ such that $(\mathbf{C}_+)_{>0} = \mathbf{C}$. Applying this construction to the semi-theory $\mathbf{P}$ (Section 5) we obtain a semi-theory $\mathbf{P}_+$ which is initial among all pointed semi-theories.

For a free pointed semi-theory $\mathbf{C}$ the algebraic theory $\bar{\mathbf{C}}$ is constructed in a similar way as in the unpointed case. A tree T representing a morphism $[n] \to [1]$ in $\bar{\mathbf{C}}$ has its non-initial edges labeled with generators of $\mathbf{C}$. However, one must allow that the labels of the initial edges can be the projection morphisms and the morphism $\iota_n \in \mathbf{C}$ which is the (unique) composition $[n] \to [0] \to [1]$. As a consequence morphisms in $\mathrm{Hom}_{\bar{\mathbf{C}}}([n],[1])$ are now equivalence classes of trees rather than trees themselves. The equivalence relation is generated by the following condition: trees T and T' are equivalent if T' is obtained by removing an initial edge of T labeled with ι_n and grafting in its place any tree S whose initial edges are all labeled with ι_n.

The composition of $(T_1,\ldots,T_m)$ representing a morphism in $\mathrm{Hom}_{\bar{\mathbf{C}}}([n],[m])$ with $S \in \mathrm{Hom}_{\bar{\mathbf{C}}}([m],[1])$ is a morphism represented by the tree obtained by grafting T_k in place of each edge of S labeled with p^m_k and relabeling with ι_n all initial edges of S which have the label ι_m.

Finally, the morphisms in $\bar{\mathbf{C}}$ with the source or target $[0]$ are determined by the conditions that $[0]$ is the initial and the terminal object in $\bar{\mathbf{C}}$, and that the composition $[n] \to [0] \to [1]$ in $\bar{\mathbf{C}}$ is represented by the tree which has only one edge, and it is labeled with ι_n.

In order to construct the resolution $\mathbf{F}_*\mathbf{C}$ for a pointed semi-theory $\mathbf{C}$ take $\mathbf{F}_0\mathbf{C}$ to be the free pointed semi-theory such that $(F_0\mathbf{C})_{>0}$ is the free category generated by the full subcategory of $\mathbf{C}$ on objects $[1],[2],\ldots$. For $k>0$ we construct $\mathbf{F}_{k+1}\mathbf{C}$ out of $\mathbf{F}_k\mathbf{C}$ in the same manner. The algebraic theory $\overline{\mathbf{F}_*\mathbf{C}}$ is obtained by applying the (pointed) completion in every simplicial dimension of $\mathbf{F}_*\mathbf{C}$.

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